We remark that for $q = 0$ the invertibility question is trivial, for $q = 1$ our result is the same as the Sain–Massey result, and our result is stronger for $q > 1$. In particular, if $D = 0$, we have

Corollary 2: The system

$$x(k+1) = A\,x(k) + B\,u(k) \tag{25}$$

$$y(k) = C\,x(k) \tag{26}$$

is invertible if and only if it is $(n - m + 1)$ − delay invertible. ∎

We also have the following strengthened corollary, the proof of which involves a trivial modification of the analogous result in [1] if we keep the above proof of Corollary 1 in mind.

Corollary 3: The system (3),(4) is invertible if and only if there is no input segment $U_{n-q+1} \neq 0$ followed by all zeroes, which produces the all zero output sequence in (3),(4) when $x_0 = 0$. ∎

Similarly, we obtain a strengthened version of the single matrix result in [1] and [3].

Theorem 2: The system (3),(4) is invertible if and only if

$$\text{rank}\,(N) = (n - q + 2)m \tag{27}$$

where N is the $(2n - q + 2)p \times (n - q + 2)m$ matrix

$$N = \begin{bmatrix} D & 0 & \cdots & 0 \\ C\,B & D & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ C\,A^{n-q}B & C\,A^{n-q-1}B & \cdots & D \\ C\,A^{n-q+1}B & C\,A^{n-q}B & \cdots & C\,B \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C\,A^{2n-q}B & C\,A^{2n-q-1}B & \cdots & C\,A^{n-1}B \end{bmatrix}. \tag{28}$$ ∎

Corollary 4: The rank condition

$$\text{rank}\,(N) = (n - q + 2)m \tag{29}$$

holds if and only if

$$\text{rank}\,(M_{n-q+1}) - \text{rank}\,(M_{n-q}) = m. \quad ∎ \tag{30}$$

We also note that in a similar manner one can obtain strengthened versions of the necessary and sufficient conditions, presented in [1] and [3], for the dual concept of functional controllability.

IV. Conclusions

In this note we have obtained a strengthened version of the necessary and sufficient conditions, derived in [1]–[3], for linear system invertibility. These results reduce the question of invertibility to a set of rank tests for certain matrices, and our strengthening of these results depends on a careful counting argument.

The question of system invertibility is important in such applications as the design of encoding-decoding systems, and has received a great deal of attention in the literature. We refer the reader to more general invertibility results in [6]–[9]. In particular the finite group system results in [6]–[8] are quite similar in flavor to the results in [1]–[3] and in this note.

References

[1] M. K. Sain and J. L. Massey, "Invertibility of linear time-invariant dynamical systems," *IEEE Trans. on Automat. Contr.*, vol. AC-14, pp. 141–149, Apr. 1969.
[2] J. L. Massey and M. K. Sain, "Inverses of linear sequential circuits," *IEEE Trans. Comput.*, vol. C-17, pp. 330–337, Apr. 1968.
[3] R. W. Brockett and M. D. Mesarovic, "The reproducibility of multivariable systems," *J. Math. Anal. Appl.*, vol. 11, pp. 548–563, July 1965.
[4] G. D. Forney, Jr., "Convolutional codes I: Algebraic structure," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 720–738, Nov. 1970.
[5] S. H. Wang and E. J. Davison, "A new invertibility criterion for linear multivariable systems," *IEEE Trans. Automat. Contr.* (Tech. Notes and Corresp.), vol. AC-18, pp. 538–539, Oct. 1973.
[6] A. S. Willsky, "Dynamical systems defined on groups: Structural properties and estimation," Ph.D. dissertation, Dept. Aeronautics and Astronautics, Mass. Inst. Technol., Cambridge, Mass., June 1973.
[7] A. S. Willsky, "Invertibility conditions for a class of finite state systems evolving on groups," *Proc. 11th Allerton Conf.*, Univ. of Illinois, Oct. 3–5, 1973.
[8] A. S. Willsky, "Invertibility of finite group homomorphic sequential systems," *Inform. Contr.*, to be published.
[9] R. R. Olson, "On the invertibility of finite state machines," Dep. Elec. Eng., Univ. of Notre Dame, Tech. Rep. EE-703, July 23, 1970.

Minimal Order Observers and Certain Singular Problems of Optimal Estimation and Control

HARRY G. KWATNY

***Abstract*—It is shown that a Riccati equation of particular structure which arises in a number of singular optimal estimation and control processes can be reduced in order. This fact leads directly to a procedure for the design of a class of minimal order observers, the structure of which can be interpreted as the limiting form of appropriate Kalman estimators with vanishing observation noise.**

I. Introduction

As might be anticipated, the theory of minimal order observers can be closely allied with certain singular problems of optimal estimation and control. This commonality is particularly striking when it is recognized that minimal order observer design can be accomplished through solution of a matrix Riccati equation which is identical in structure to those arising in singular optimal regulator problems and which admits a reduction in order.

It is known that the problem of minimal order observer design for an nth order, completely observable system with r independent outputs can be conveniently solved by solution of an $(n - r) \times (n - r)$ dimension matrix Riccati equation [1].

In what follows it is shown that the required Riccati equation can be derived through reduction of a larger $n \times n$ Riccati equation and that, in appropriate circumstances, observers designed in this way are limiting forms of Kalman estimators for vanishing observation noise in the sense of Friedland [2]. Furthermore, it is observed that certain Riccati equations obtained by Friedland [2] and Moylan and Moore [3] for singular optimal regulator problems are structured identically to that obtained for the observer design problem and can be reduced in order. Certain problems of estimating the state of a linear dynamical system from observation of outputs corrupted by correlated noise are duals of these singular regulator problems and consequently can be solved by identical procedures.

II. Order Reduction of a Class of Riccati Equations

Let C be an $r \times n$ matrix of full rank and let C_0^* denote a right inverse of C.

Let P be an $n \times n$ symmetric matrix which satisfies the relation

$$CP = 0 \tag{1}$$

as well as the algebraic Riccati equation

$$PA'[I - C_0^*C]' + [I - C_0^*C]AP - PA'J'S^{-1}JAP + M = 0 \tag{2}$$

where $S > 0$ is a symmetric $r \times r$ matrix, $M \geq 0$ is a symmetric $n \times n$ matrix which has the property

$$CM = 0. \tag{3}$$

J is an $r \times n$ matrix which will be required to satisfy a controllability condition given below. A method will be given for obtaining P satisfying both (1) and (2) by solving a Riccati equation of dimension $(n - r) \times (n - r)$.

Manuscript received July 13, 1973; revised November 15, 1973.
The author is with the College of Engineering, Drexel University, Philadelphia, Pa.

Define another right inverse of C

$$C^* = C_0^* + PA'J'S^{-1} \tag{4}$$

and also the matrices Λ_0, Ψ_0, Θ_0 and Λ, Ψ, Θ as follows:

Λ_0, Λ are $(n-r) \times n$ matrices composed, respectively, of $n-r$ linearly independent rows of $[I - C_0^*C]$ and $[I - C^*C]$

$$\Psi_0 = \left[\begin{array}{c} C \\ \hline \Lambda_0 \end{array}\right], \quad \Psi = \left[\begin{array}{c} C \\ \hline \Lambda \end{array}\right] \tag{5}$$

$$\Psi_0^{-1} = \Theta_0 = \left[\begin{array}{c|c} \Theta_{01} & \Theta_{02} \end{array}\right], \quad \Psi^{-1} = \Theta = \left[\begin{array}{c|c} \Theta_1 & \Theta_2 \end{array}\right].$$
$$\qquad \begin{array}{cc} \leftrightarrow & \longleftrightarrow \\ r & n-r \end{array} \qquad\qquad \begin{array}{cc} \leftrightarrow & \longleftrightarrow \\ r & n-r \end{array}$$

The controllability requirement imposed on J can now be stated. It is assumed that J is specified such that the pair $(\Theta_{02}'A'\Lambda_0')$, $(\Theta_{02}'A'J')$ is completely controllable. It is easily shown that this condition is satisfied if A',J' is a controllable pair.

The following observation facilitates the construction of Λ (or Λ_0) and Ψ^{-1} (or Ψ_0^{-1}).

Observation: The matrix C can always be arranged so that $C = [C_1 \; C_2]$ where C_2 is an $r \times r$ dimensional nonsingular matrix in which case Λ and Ψ^{-1} can be obtained as follows:

$$\Lambda = [I_{n-r} - C_1^*C_1 | - C_1^*C_2], \quad C^* = \left[\begin{array}{c} C_1^* \\ \hline C_2^* \end{array}\right] \begin{array}{l} \updownarrow \; n-r \\ \updownarrow \; r \end{array} \tag{6a}$$

$$\Psi^{-1} = \left[\begin{array}{c|c} C_1^* & I_{n-r} \\ \hline C_2^* & -C_2^{-1}C_1 \end{array}\right]. \tag{6b}$$

The main result is given in the following theorem which reduces the problem of solving (1) and (2) to that of solving a reduced order Riccati equation.

Theorem 1: P is a solution of (1) and (2) iff

$$P = (\Psi_0^{-1}) \operatorname{diag}(0_r, \rho_{22})(\Psi_0^{-1})' \tag{7a}$$

where ρ_{22} is a solution of the $(n-r) \times (n-r)$ Riccati equation

$$\rho_{22}(\Theta_{02}'A'\Lambda_0') + (\Lambda_0 A\Theta_{02})\rho_{22} - \rho_{22}(JA\Theta_{02})'S^{-1}(JA\Theta_{02})\rho_{22} + (\Lambda_0 M\Lambda_0') = 0. \tag{7b}$$

Proof of Theorem 1: Postmultiply (1) by Ψ_0' and note that $C\Psi_0^{-1} = [I_r \; 0_{r\times(n-r)}]$ to obtain

$$0 = CP\Psi_0' = C\Psi_0^{-1}\Psi_0 P\Psi_0' = [I_n | 0_{r\times(n-r)}]\rho \tag{8}$$

where

$$\Psi_0 P\Psi_0' \triangleq \rho = \left[\begin{array}{c|c} \rho_{11} & \rho_{12} \\ \hline \rho_{12}' & \rho_{22} \end{array}\right] \begin{array}{l} \updownarrow \; r \\ \updownarrow \; n-r. \end{array} \tag{9}$$
$$\begin{array}{cc} \longleftrightarrow & \longleftrightarrow \\ n & n-r \end{array}$$

From (8) it is clear that

$$\rho_{11} = 0_r, \quad \rho_{12} = 0_{r\times(n-r)}. \tag{10}$$

Only ρ_{22} remains to be determined. The following formulae will be useful:

$$(I - C_0^*C) = \Psi_0^{-1}\bar{E}\Psi_0, \text{ where } \bar{E} = \operatorname{diag}(0_r, I_{n-r}) \tag{11a}$$

$$\Psi_0(I - C_0^*C)A\Psi_0^{-1} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \Lambda_0 A\Theta_{01} & \Lambda_0 A\Theta_{02} \end{array}\right] \tag{11b}$$

$$\Psi_0 M\Psi_0' = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & (\Lambda_0 M\Lambda_0') \end{array}\right] \begin{array}{l} \updownarrow \; r \\ \updownarrow \; n-r. \end{array} \tag{11c}$$
$$\begin{array}{cc} \leftrightarrow & \longleftrightarrow \\ r & n-r \end{array}$$

Premultiply (2) by Ψ_0 and postmultiply by Ψ_0' and make use of the definition (9) to obtain

$$\rho\{\Psi_0(I - C_0^*C)A\Psi_0^{-1}\}' + \{\Psi_0(I - C_0^*C)A\Psi_0^{-1}\}\rho - \rho[(\Psi_0^{-1})'A'J'S^{-1}JA\Psi_0^{-1}]\rho + \Psi_0 M\Psi_0' = 0. \tag{12}$$

Making use of (10) and (11) in (12) it is readily found that ρ_{22} must satisfy (7).

An important observation is stated in the following corollary.

Corollary: Let ρ_{22} be the maximal solution of (7) and $J = C$. Then $\Lambda A\Theta_2$ is asymptotically stable.

Proof: Consider the regulator interpretation of (7b). That is, $\Theta_{02}'A'\Lambda_0'$ is the open-loop system matrix, $\Theta_{02}'A'J'$ is the gain matrix and S and $\Lambda_0 M\Lambda_0'$ the control and state weighting matrices, respectively. Since the pair $\Theta_{02}'A'\Lambda_0'$ and $\Theta_{02}'A'J'$ is completely controllable, then with ρ_{22} the maximal solution of (7b), the closed-loop system matrix

$$\tilde{A}_{22} = \Theta_{02}'A'\Lambda_0' - \Theta_{02}'A'J'S^{-1}JA\Theta_{02}\rho_{22} \tag{13}$$

is asymptotically stable. Moreover, simple computation shows that

$$(\Psi_0')^{-1}[A'\{I - C^*C\}]\Psi_0' = \left[\begin{array}{c|c} 0_{n\times r} & \begin{array}{c} A_{12} \\ \hline A_{22} \end{array} \end{array}\right] \tag{14}$$

and

$$(\Psi')^{-1}[A'\{I - C^*C\}']\Psi' = \left[\begin{array}{c|c} 0_{n\times r} & \begin{array}{c} \Theta_1 A'\Lambda' \\ \hline \Theta_2'A'\Lambda' \end{array} \end{array}\right]. \tag{15}$$

Thus, these two matrices are similar and have the same eigenvalues. Consequently, $\Theta_2'A'\Lambda'$ has the same eigenvalues as $\tilde{A}_{22}$ and is asymptotically stable.

III. Minimal Order Observer Design

Consider a linear time-invariant plant

$$\begin{aligned} x &= Ax + Bu \\ y &= Cx \end{aligned} \tag{16}$$

where x, u, y are vectors of order n, m, $r \le n$, respectively. It is assumed that (AC) is completely observable and that C is of full rank. A p-dimensional linear system

$$\begin{aligned} \dot{z} &= Fz + Gy + Hu \\ w &= W\left[\begin{array}{c} z \\ \hline y \end{array}\right] \end{aligned} \tag{17}$$

where z, w are vectors of order p, n, respectively, and W is an $n \times (p + r)$ matrix is called an observer or an asymptotic state estimator for the system (16) if

$$\lim_{t\to\infty} \{x(t) - w(t)\} = 0_n. \tag{18}$$

It is well known [4]–[7] that (17) is an observer for (16) if and only if:

1) F is asymptotically stable ($\mathrm{Re}\lambda_i(F) < 0$, $i = 1,2,\cdots,p$)
2) There exists a $p \times n$ matrix T such that
$$TA - FT = GC \tag{19}$$
3) $$H = TB$$
4) the matrix W satisfies
$$I_n = W\left[\begin{array}{c} T \\ \hline C \end{array}\right].$$

Furthermore, the minimum dimension of p is $p = n - r$ and a minimal order observer exists for (16).

The fundamental problem of observer theory is the selection of the parameters F, G, H, and W such that (19) is satisfied. This is usually accomplished by selecting T to satisfy (19b) and so that the eigenvalues of F are assigned specified values. Several algorithms have been proposed [5]–[7] for designing observers in this manner and center on placing the observed system in a suitable canonical form.

An alternative procedure proposed by Johnson [1] and which can be considerably less cumbersome to apply is stated below in terms of the parameters defined in the previous section.

Theorem 2: An observer for (16) is given by (17) with the parameters defined as follows and $J = C$:

$$F = \Lambda A\Theta_2,\ G = \Lambda AC^*,\ H = \Lambda B,\ W = [\Theta_2 \vdots C^*],\ T = \Lambda. \quad (20)$$

Proof: It is necessary to show that the conditions (19) are satisfied. By the Corollary to Theorem 1, $F = \Lambda A\Theta_2$ is asymptotically stable and (19a) is satisfied. Condition (19b) is verified by computation.

$$\begin{aligned} TA - FT &= \Lambda A - \Lambda A\Theta_2\Lambda \\ &= \Lambda A - \Lambda A[I - C^*C] = GC. \end{aligned}$$

(19c) is obvious and (19d) is again verified by computation.

$$W\begin{bmatrix} T \\ \hline C \end{bmatrix} = [\Theta_2 \vdots C^*]\begin{bmatrix} \Lambda \\ \hline C \end{bmatrix} = \Theta_2\Lambda + C^*C = I_n.$$

IV. Singular Estimation and Control

It will now be shown how the results of the previous sections relate to certain singular problems of optimal estimation and control. In particular, if the following choices are made:

$$\begin{aligned} J &= C, \\ S &= C\Sigma C',\ \Sigma \geq 0 \text{ such that } S^{-1} \text{ exists} \\ C^* &= \Sigma C'(C\Sigma C')^{-1} \\ M^0 &= \Sigma - \Sigma C'(C\Sigma C')^{-1}C\Sigma, \end{aligned} \quad (21)$$

then the parameters of Theorem 2 yield an observer which is the limiting form of the Kalman estimator [2] for the system

$$\begin{aligned} \dot{x} &= Ax + Bu + \sigma, \\ y &= Cx + \eta, \end{aligned} \quad (22)$$

as $\text{cov}(\eta) \to 0$ where σ, η are zero mean, white Gaussian signals with $\text{cov}(\sigma) = \Sigma$.

It is also noted that the singular problem of optimally regulating the completely controllable process

$$\dot{x} = Ax + Bu$$

with respect to the performance index

$$V = \int_0^\infty x'Qx\, dt$$

is solved by the control law [3]

$$u^* = -(B'QB)^{-1}(B'A'P + B'Q)Ax$$

provided that $(B'QB)$ is nonsingular and where P satisfies (1), (2) with the following replacements

$$\begin{aligned} C_0^* &\to QB(B'QB)^{-1}B' \\ C = J &\to B' \\ A &\to A' \\ S &\to (B'QB) \\ M &\to Q - QB(B'QB)^{-1}B'Q. \end{aligned}$$

Accordingly, P can be obtained via solution of a reduced order Riccati equation as described in Theorem 1.

IV. Conclusions

It has been shown that an $n \times n$ dimensional Riccati equation of particular structure can be reduced to an $(n - r) \times (n - r)$ dimensional equation. It was noted that such equations arise in certain singular problems of optimal estimation and control. As a result it is possible to identify a known class of minimal order observers as limiting forms of Kalman estimators for appropriate optimal estimation problems with vanishing observation noise.

References

[1] C. D. Johnson, "Accommodation of external disturbances in linear regulator and servomechanism problems," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 635–64, Dec. 1971.

[2] B. Friedland, "Limiting forms of optimum stochastic linear regulators," *Trans. ASME*, series G, vol. 93, Sept. 1971.

[3] P. J. Moylan and J. B. Moore, "Generalizations of singular optimal control theory," *Automatica*, vol. 7, pp. 591–598, Sept. 1971.

[4] D. G. Luenberger, "Observers for multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 190–197, Apr. 1966.

[5] W. A. Wolovich, "On the state estimation of observable systems," in *Proc. 9th Joint Automatic Control Conf.*, 1968, pp. 210–222.

[6] Y. O. Yüksel and J. J. Bongiorno, Jr., "Observers for linear multivariable systems with applications," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 603–613, Dec. 1971.

[7] T. E. Fortmann and D. Williamson, "Design of low-order observers for linear feedback control laws," *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 301–308, June 1972.

On the Cancellation of Multivariable System Zeros by State Feedback

W. A. WOLOVICH

***Abstract*—The fact that all of the defined zeros of a proper linear multivariable system can be cancelled by an appropiate linear state variable feedback control law is constructively established.**

Introduction

In the case of a scalar (single input/output) linear system characterized by a rational transfer function, $t(s) = r(s)/p(s)$, it is well known that linear state variable feedback (lsvf) can be used to completely and arbitrarily assign all (n) poles of the closed-loop transfer function, $t_f(s) = r(s)/p_f(s)$; i.e., the zeros of $p_f(s)$. Therefore, if $t(s)$ is proper (if the degree of $r(s)$ does not exceed the degree of $p(s)$) lsvf can be used to "completely cancel" all of the zeros of $[r(s)]$ the system. The primary purpose of this note is to formally extend this result to include linear multivariable systems,[1] and we begin by reviewing some consequences of lsvf compensation in the multivariable case.

State Feedback Preliminaries

It is well known [1],[2] that the $(p \times m)$ transfer matrix, $T(s)$, of a linear, time-invariant, multivariable system can be factored as the product

$$T(s) = R(s)P(s)^{-1} \quad (1)$$

where $R(s)$ and $P(s)$ are relatively right prime polynomial matrices in the Laplace operator s, and $P(s)$ is *column proper* (defined as the condition that the real matrix consisting of the coefficients of the highest degree s-term or terms in each column of $P(s)$ be of full rank). Furthermore, it is also well known [3] that if $T(s)$ is a proper transfer matrix, then the degree of each column of $R(s)$ will be no greater than the degree of each corresponding column of $P(s)$; a relation which we succinctly express as

$$\partial_c[R(s)] \leqslant \partial_c[P(s)]. \quad (2)$$

It is of interest to note that (2) holds for any factorization, $R(s)P(s)^{-1}$, of the proper transfer matrix $T(s)$; i.e., whether $P(s)$ is column proper or not.

Manuscript received January 7, 1974. This work was supported in part by the National Science Foundation Grant NSF GK-40493X and the Air Force Office of Scientific Research Grant AFOSR 71-2078.

The author is with the Division of Engineering and the Center for Dynamical Systems, Brown University, Providence, R.I. 02912.

[1] The fact that lsvf can be used to completely cancel all of the zeros of a linear multivariable system characterized by a proper rational transfer matrix has already been noted, [1],[2] but not formally established.